PRACTICE SET FOR MIDTERM 1

Problem 1

Find the natural domain and the horizontal/vertical asymptotes of the following functions

(1) $\quad f(x) = \frac{x+1}{x^2}.$

Domain: $x^2 \neq 0$, so $x \neq 0$. In interval notation

$$]-\infty, 0[\cup]0, +\infty[.$$

Asymptotes:

$$\lim_{x\to-\infty} \frac{x+1}{x^2} = \lim_{x\to-\infty} \frac{x(1+\frac{1}{x})}{x^2} = \lim_{x\to-\infty} \frac{1+\frac{1}{x}}{x} = 0 \ \rightarrow y = 0 \text{ HOR ASYMPTOTE (at } -\infty)$$

$$\lim_{x\to+\infty} \frac{x+1}{x^2} = \lim_{x\to+\infty} \frac{x(1+\frac{1}{x})}{x^2} = \lim_{x\to+\infty} \frac{1+\frac{1}{x}}{x} = 0 \ \rightarrow y = 0 \text{ HOR ASYMPTOTE (at } +\infty)$$

$$\lim_{x\to0^-} \frac{x+1}{x^2} = +\infty \ \rightarrow x = 0 \text{ VERT ASYMPTOTE}$$

$$\lim_{x\to0^+} \frac{x+1}{x^2} = +\infty \ \rightarrow x = 0 \text{ VERT ASYMPTOTE}$$

(2) $\quad f(x) = e^{\frac{1}{x^2-9}}.$

Domain: $x^2 - 9 \neq 0$, so $x \neq \pm 3$. In interval notation

$$]-\infty, -3[\cup]-3, 3[\cup]3, +\infty[.$$

Asymptotes

$$\lim_{x\to-\infty} \frac{1}{x^2-9} = 0 \ \Rightarrow \lim_{x\to-\infty} e^{\frac{1}{x^2-9}} = e^0 = 1 \ \rightarrow y = 1 \text{ HOR ASYMPTOTE (at } -\infty)$$

$$\lim_{x\to+\infty} \frac{1}{x^2-9} = 0 \ \Rightarrow \lim_{x\to+\infty} e^{\frac{1}{x^2-9}} = e^0 = 1 \ \rightarrow y = 1 \text{ HOR ASYMPTOTE (at } +\infty)$$

$$\lim_{x\to-3^-} \frac{1}{x^2-9} = +\infty \ \Rightarrow \lim_{x\to-3^-} e^{\frac{1}{x^2-9}} = +\infty \ \rightarrow x = -3 \text{ VERT ASYMPTOTE}$$

$$\lim_{x\to-3^+} \frac{1}{x^2-9} = -\infty \ \Rightarrow \lim_{x\to-3^+} e^{\frac{1}{x^2-9}} = 0$$

$$\lim_{x\to3^-} \frac{1}{x^2-9} = -\infty \ \Rightarrow \lim_{x\to3^-} e^{\frac{1}{x^2-9}} = 0$$

$$\lim_{x\to3^+} \frac{1}{x^2-9} = +\infty \ \Rightarrow \lim_{x\to3^+} e^{\frac{1}{x^2-9}} = +\infty \ \rightarrow x = 3 \text{ VERT ASYMPTOTE}$$

(3) $\qquad f(x) = \text{arctg}\left(\dfrac{x^3+4}{x^3}\right).$

Domain: $x^3 \neq 0$, so $x \neq 0$. In interval notation

$$]-\infty, 0[\cup]0, +\infty[.$$

Asymptotes:

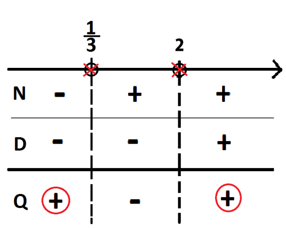
$$\lim_{x\to-\infty} \frac{x^3+4}{x^3} = \lim_{x\to-\infty} 1 + \frac{4}{x^3} = 1 \Rightarrow \lim_{x\to-\infty} \text{arctg}\left(\frac{x^3+4}{x^3}\right) = \text{arctg}(1) = \frac{\pi}{4}$$
$$\to y = \frac{\pi}{4} \text{ HOR ASYMPTOTE (at } -\infty)$$
$$\lim_{x\to+\infty} \frac{x^3+4}{x^3} = \lim_{x\to+\infty} 1 + \frac{4}{x^3} = 1 \Rightarrow \lim_{x\to-\infty} \text{arctg}\left(\frac{x^3+4}{x^3}\right) = \text{arctg}(1) = \frac{\pi}{4}$$
$$\to y = \frac{\pi}{4} \text{ HOR ASYMPTOTE (at } +\infty)$$
$$\lim_{x\to0^-} \frac{x^3+4}{x^3} = -\infty \Rightarrow \lim_{x\to0^-} \text{arctg}\left(\frac{x^3+4}{x^3}\right) = -\frac{\pi}{2}$$
$$\lim_{x\to0^+} \frac{x^3+4}{x^3} = +\infty \Rightarrow \lim_{x\to0^+} \text{arctg}\left(\frac{x^3+4}{x^3}\right) = \frac{\pi}{2}$$

so no vertical asymptotes.

(4) $\qquad f(x) = \ln\left(\dfrac{3x-1}{x-2}\right).$

Domain: $\frac{3x-1}{x-2} > 0$.

$$N: 3x-1>0 \quad \Rightarrow \quad x > \frac{1}{3}$$
$$D: x-2>0 \quad \Rightarrow \quad x>2$$



In interval notation

$$]-\infty, \frac{1}{3}[\cup]2, +\infty[.$$

Asymptotes:

$$\lim_{x\to-\infty}\frac{3x-1}{x-2}=\lim_{x\to-\infty}\frac{x(3-\frac{1}{x})}{x(1-\frac{2}{x})}=3\Rightarrow\lim_{x\to-\infty}\ln\left(\frac{3x-1}{x-2}\right)=\ln(3)$$

$$\to y=\ln(3)\text{ HOR ASYMPTOTE (at }-\infty)$$

$$\lim_{x\to+\infty}\frac{3x-1}{x-2}=\lim_{x\to+\infty}\frac{x(3-\frac{1}{x})}{x(1-\frac{2}{x})}=3\Rightarrow\lim_{x\to+\infty}\ln\left(\frac{3x-1}{x-2}\right)=\ln(3)$$

$$\to y=\ln(3)\text{ HOR ASYMPTOTE (at }+\infty)$$

$$\lim_{x\to\frac{1}{3}^-}\frac{3x-1}{x-2}=0\Rightarrow\lim_{x\to\frac{1}{3}^-}\ln\left(\frac{3x-1}{x-2}\right)=-\infty\ \to x=\frac{1}{3}\text{ VERT ASYMPTOTE}$$

$$\lim_{x\to2^+}\frac{3x-1}{x-2}=+\infty\Rightarrow\lim_{x\to2^+}\ln\left(\frac{3x-1}{x-2}\right)=+\infty\ \to x=2\text{ VERT ASYMPTOTE}$$

(5) $f(x)=\dfrac{\cos(x^2)}{x^2+1}$.

Domain: $x^2+1\neq0$. Since x^2 is always positive, x^2+1 is always different than 0, and the domaoin is $\mathbb{R}$. In interval notation

$$]-\infty,+\infty[.$$

Asymptotes: Note that $-1\leq\cos(x^2)\leq1$ for every $x\in\mathbb{R}$. Therefore

$$-\frac{1}{x^2+1}\leq\frac{\cos(x^2)}{x^2+1}\leq\frac{1}{x^2+1}.$$

Now, since

$$\lim_{x\to-\infty}-\frac{1}{x^2+1}=0\quad\text{and}\quad\lim_{x\to-\infty}\frac{1}{x^2+1}=0,$$

by the squeeze theorem

$$\lim_{x\to-\infty}\frac{\cos(x^2)}{x^2+1}=0\quad\to\quad y=0\quad\text{HOR ASYMPTOTE (at }-\infty).$$

Similarly, by the squeeze theorem

$$\lim_{x\to+\infty}\frac{\cos(x^2)}{x^2+1}=0\quad\to\quad y=0\quad\text{HOR ASYMPTOTE (at }+\infty).$$

Problem 2

Compute the following limits if they exist or prove that they don't exist

(6) $$\lim_{x\to+\infty}\frac{3e^{4x}+1}{e^x-e^{4x}}=\lim_{x\to+\infty}\frac{e^{4x}(3+\frac{1}{e^{4x}})}{e^{4x}(\frac{1}{e^{3x}}-1)}=\lim_{x\to+\infty}\frac{3+\frac{1}{e^{4x}}}{\frac{1}{e^{3x}}-1}=-3.$$

(7) $$\begin{aligned}\lim_{x\to0}\frac{2-\sqrt{4+x^2}}{9x^2}&=\lim_{x\to0}\frac{2-\sqrt{4+x^2}}{9x^2}\cdot\frac{2+\sqrt{4+x^2}}{2+\sqrt{4+x^2}}=\lim_{x\to0}\frac{4-4-x^2}{9x^2(2+\sqrt{4+x^2})}\\&=\lim_{x\to0}-\frac{1}{9(2+\sqrt{4+x^2})}=-\frac{1}{9\cdot4}\end{aligned}$$

(8) $$\begin{aligned}\lim_{x\to0}\frac{\frac{10}{3+2x}-\frac{10}{3}}{x^2-2x}&=\lim_{x\to0}\frac{\frac{30-30-20x}{3(3+2x)}}{x(x-2)}=\lim_{x\to0}\frac{-20x}{3(3+2x)}\cdot\frac{1}{x(x-2)}=\lim_{x\to0}\frac{-20}{3(3+2x)(x-2)}\\&=\frac{-20}{3\cdot3\cdot(-2)}=\frac{10}{9}\end{aligned}$$

(9) $$\lim_{x\to-\infty}\cos(4x)\cdot e^{-x^2}.$$

Note that $-1\le\cos(4x)\le1$ for every $x\in\mathbb{R}$. Therefore

$$-e^{-x^2}\le\cos(4x)\cdot e^{-x^2}\le e^{-x^2}.$$

Now, since $\lim_{x\to-\infty}(-x^2)=-\infty$ we have

$$\lim_{x\to-\infty}-e^{-x^2}=0\quad\text{and}\quad\lim_{x\to-\infty}e^{-x^2}=0.$$

Therefore, by the squeeze theorem

$$\lim_{x\to-\infty}\cos(4x)\cdot e^{-x^2}=0.$$

(10) $$\lim_{x\to+\infty}\sin(x)\cdot\frac{x}{x^2-2}.$$

Similarly to the previous problem we have $-1\le\sin(x)\le1$ for every $x\in\mathbb{R}$. Therefore

$$-\frac{x}{x^2-2}\le\sin(x)\cdot\frac{x}{x^2-2}\le\frac{x}{x^2-2}.$$

Now we can compute

$$\lim_{x\to\infty}-\frac{x}{x^2-2}=\lim_{x\to\infty}-\frac{x}{x^2(1-\frac{2}{x^2})}=\lim_{x\to\infty}-\frac{1}{x}\cdot\frac{1}{(1-\frac{2}{x^2})}=0$$
$$\lim_{x\to\infty}\frac{x}{x^2-2}=\lim_{x\to\infty}\frac{x}{x^2(1-\frac{2}{x^2})}=\lim_{x\to\infty}\frac{1}{x}\cdot\frac{1}{(1-\frac{2}{x^2})}=0.$$

Therefore by the squeeze theorem we have

$$\lim_{x\to+\infty}\sin(x)\cdot\frac{x}{x^2-2}=0.$$

(11) $$\lim_{x\to0}\frac{|x^3|}{x^3-x}.$$

The function $|x^3|$ is piecewise defined:

$$|x^3|=\begin{cases}x^3 & \text{if}\quad x\ge0\\-x^3 & \text{if}\quad x<0\end{cases}$$

therefore we compute the sided limits:

$$\lim_{x\to 0^+} \frac{|x^3|}{x^3 - x} = \lim_{x\to 0^+} \frac{x^3}{x(x^2-1)} = \lim_{x\to 0^+} \frac{x^2}{x^2-1} = 0$$

$$\lim_{x\to 0^-} \frac{|x^3|}{x^3 - x} = \lim_{x\to 0^+} \frac{-x^3}{x(x^2-1)} = \lim_{x\to 0^+} \frac{-x^2}{x^2-1} = 0$$

and in conclusion

$$\lim_{x\to 0} \frac{|x^3|}{x^3 - x} = 0$$

.

(12) $\displaystyle \lim_{x\to -1} \frac{2x+2}{|x+1|}$

The function $|x+1|$ is piecewise defined:

$$|x+1| = \begin{cases} x+1 & \text{if} \quad x \geq -1 \\ -(x+1) & \text{if} \quad x < -1 \end{cases}$$

therefore we compute the sided limits:

$$\lim_{x\to -1^+} \frac{2x+2}{|x+1|} = \lim_{x\to -1^+} \frac{2(x+1)}{x+1} = \lim_{x\to -1^+} \frac{2}{1} = 2$$

$$\lim_{x\to -1^-} \frac{2x+2}{|x+1|} = \lim_{x\to -1^-} \frac{-2(x+1)}{x+1} = \lim_{x\to -1^-} \frac{-2}{1} = -2$$

and we conclude that

$$\lim_{x\to -1} \frac{2x+2}{|x+1|} \quad \text{doesn't exist.}$$

(13)

$$\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{7x} = \lim_{x\to -\infty} \frac{\sqrt{x^2(1+\frac{1}{x^2})}}{7x} = \lim_{x\to -\infty} \frac{-x\cdot\sqrt{1+\frac{1}{x^2}}}{7x} = \lim_{x\to -\infty} \frac{-\sqrt{1+\frac{1}{x^2}}}{7} = -\frac{1}{7}.$$

Problem 3

Are the following functions continuous? If not classify the type of discontinuity they exhibit.

(14) $\quad f(x) = \sqrt{x^6 + x^4 + 2}.$

Note that $x^6 + x^4 + 2$ is always positive, so the domain is $\mathbb{R}$.
As a composition of a square root and a polynomial (which are continuous functions) f is continuous everywhere.

(15) $\quad f(x) = |x^2 - 4|.$

As a composition of the absolute value function and a polynomial (which are continuous) f is continuous everywhere.

(16) $$f(x) = \begin{cases} \frac{x+1}{x-2} & \text{for} \quad x > 2 \\ x^3 - 1 & \text{for} \quad x \leq 2 \end{cases}$$

Note that the function is continuous at every point different than 2. We need to check $x = 2$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^3 - 1) = 7$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x+1}{x-2} = +\infty$$

so the function has an INFINITE discontinuity at $x = 2$.

(17) $$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for} \quad x \neq 0 \\ 1 & \text{for} \quad x = 0 \end{cases}$$

As a composition of continuous functions, f is continuous at every point different than 0. We need to check $x = 0$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-\frac{1}{x^2}} = 0 \quad \text{because} \quad \lim_{x\to 0^-} -\frac{1}{x^2} = -\infty$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{-\frac{1}{x^2}} = 0 \quad \text{because} \quad \lim_{x\to 0^+} -\frac{1}{x^2} = -\infty$$
$$f(0) = 1$$

so the function has a REMOVABLE discontinuity at $x = 0$.

Do the following equations admit any real solutions?

(18) $\quad \ln(x - 1) + \ln(x) = 1.$

This equation can be solved explicitly.
Domain (both logarithms must exist at the same time):

$$x > 1$$
$$x > 0$$

so we get the domain by taking the intersection, which is $x > 0$. Now we can apply properties of logarithms to solve it:

$$\ln(x(x-1)) = 1$$
$$\ln(x^2 - x) = \ln(e)$$
$$x^2 - x = e$$
$$x^2 - x - e = 0$$

and using the quadratic formula we get

$$x_1, x_2 = \frac{1 \pm \sqrt{1+4e}}{2}$$

but only $x_1 = \frac{1+\sqrt{1+4e}}{2}$ is acceptable (because $\sqrt{1+4e} > 1$ so $1 - \sqrt{1+4e} < 0$).

(19) $x^5 - x = 2.$

We can't solve this explicitly, so rewrite it in the form

$$x^5 - x - 2 = 0.$$

Consider the function $f(x) = x^5 - x - 2$, which is CONTINUOUS for every $x \in \mathbb{R}$.
Note that $f(0) = -2 < 0$ and $f(2) = 2^5 - 2 - 2 > 0$, therefore by the Intermediate Value Theorem the equation has at least one solution in the interval $[0, 2]$.

(20) $\text{arctg}(x) = x^3 - x.$

Simply note that $x = 0$ is a solution of the equation.

(21) $e^x + x^2 + 2 = 0.$

Note that $e^x + x^2 + 2$ is always greater than 2, in particular it can never be 0. So the equation doesn't admit any real solution ($\nexists x \in \mathbb{R}$).

Problem 4

Compute the derivatives of the following functions

(22)
$$f(x) = (x^3 - 3x^2 + 1)\cos(3x^2)$$
$$f'(x) = (3x^2 - 6x)\cos(3x^2) + (x^3 - 3x^2 + 1)(-\sin(3x^2))6x$$

(23)
$$f(x) = \frac{\operatorname{arctg}(x^2)}{e^{-x}} = e^x \operatorname{arctg}(x^2)$$
$$f'(x) = e^x \operatorname{arctg}(x^2) + e^x \frac{1}{1+x^4}(2x)$$

(24)
$$f(x) = \sqrt[4]{x^6 - 2} = (x^6 - 2)^{\frac{1}{4}}$$
$$f'(x) = \frac{1}{4}(x^6 - 2)^{\frac{-3}{4}} 6x^5$$

(25)
$$f(x) = \arcsin\left(\frac{3}{x^3}\right) = \arcsin(3x^{-3})$$
$$f'(x) = \frac{1}{\sqrt{1 - (3x^{-3})^2}}(-9x^{-4})$$

(26)
$$f(x) = \sin^5(3x) = (\sin(3x))^5$$
$$f'(x) = 5(\sin(3x))^4 \cos(3x) \cdot 3$$

(27)
$$f(x) = (x^5 + 1)^{2x-1} = e^{\ln((x^5+1)^{2x-1})} = e^{(2x-1)\ln(x^5+1)}$$
$$f'(x) = e^{(2x-1)\ln(x^5+1)} \left(2\ln(x^5 + 1) + (2x - 1)\frac{1}{x^5 + 1} \cdot 5x^4\right)$$

Problem 5

Find $\frac{dy}{dx}$ for the functions implicitly defined by the following equations

(28) $x^2 = y^3 - 2x.$

$$2x = 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} - 2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x+2}{3y^2}.$$

(29) $\cos(y) = xy^2 + 2$

$$-\sin(y)\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} + \sin(y)\frac{\mathrm{d}y}{\mathrm{d}x} = -y^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}(2xy + \sin(y)) = -y^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y^2}{2xy + \sin(y)}.$$

(30) $\ln(xy) = x + y$

$$\frac{1}{xy}\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{1}{x} + \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{1}{y} - 1\right) = 1 - \frac{1}{x}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - \frac{1}{x}}{\frac{1}{y} - 1}$$

(31) $\arcsin(y^2) = e^{x+y}$

$$\frac{1}{\sqrt{1-y^4}} \cdot 2y\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x+y}\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$
$$\frac{2y}{\sqrt{1-y^4}}\frac{\mathrm{d}y}{\mathrm{d}x} - e^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x+y}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^{x+y}}{\frac{2y}{\sqrt{1-y^4}} - e^{x+y}}$$

Problem 6

32) Determine the equation of the tangent line to the graph of the function $f(x) = \ln(x)$ at $x = 1$.

$f'(x) = \frac{1}{x}$.

Slope of tangent line at $x = 1$ is given by $f'(1) = 1$.

The point is given by $x = 1$, $y = \ln(1) = 0$.

Therefore the equation of the tangent is $y = x - 1$.

Is there a point where the tangent is parallel to the line $y = 3x - 1$?

Being parallel to a line means having the same slope. Therefore we are looking for points x where $f'(x) = 3$. This gives the equation $\frac{1}{x} = 3$ which has solution $x = \frac{1}{3}$. Therefore the tangent to the graph of $y = \ln(x)$ is parallel to the line $y = 3x - 1$ at the point $(1/3, \ln(1/3))$.

Is there a point where the tangent is horizontal?

Being horizontal means slope $= 0$. Therefore we need to solve

$$\frac{1}{x} = 0$$

which is impossible. So there are no points where the tangent is horizontal.

33) Determine the equation of the tangent line to the ellipse of horizontal semi-axis 3 and vertical semi-axis 4 at the point $(\sqrt{5}, \frac{8}{3})$.

The equation of such ellipse is

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

To find the tangent at $(\sqrt{5}, \frac{8}{3})$ we need to find $\frac{dy}{dx}$ at $x = \sqrt{5}$, $y = \frac{8}{3}$. We use implicit differentiation:

$$\frac{2}{9}x + \frac{2}{16}y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{9} \cdot \frac{16}{y} = -\frac{\sqrt{5}}{9} \cdot \frac{16 \cdot 3}{8} = -\frac{2}{3}\sqrt{5}.$$

Therefore the equation of the tangent is $y - \frac{8}{3} = -\frac{2}{3}\sqrt{5}(x - \sqrt{5})$.

Are there any points where the tangent to the ellipse is horizontal?

This means

$$\frac{dy}{dx} = 0 \quad \text{so} \quad -\frac{x}{9} \cdot \frac{16}{y} = 0$$

which has solution $x = 0$. This corresponds to 2 points on the ellipse, namely $(0, 4)$ and $(0, -4)$.

34)Is the line $y = -x + 2$ tangent to the graph of the function $y = \ln(x^2 + 1)$?

The given line has slope equal to -1.

First let's see if there is any point where $f'(x) = -1$.

$f(x) = \ln(x^2 + 1)$ so $f'(x) = \frac{2x}{x^2+1}$.

Solve the equation

$$\frac{2x}{x^2 + 1} = -1$$

$$2x = -x^2 - 1$$

$$x^2 + 2x + 1 = 0$$

$$(x + 1)^2 = 0$$

$$x = -1.$$

Now let's find the tangent line at $x = -1$. The y coordinate of the point on the graph is $y = \ln(1+1) = \ln(2)$, so the tangent line is

$$y - \ln(2) = -(x+1)$$

which is clearly different than $y = -x + 2$. In conclusion $y = -x + 2$ is not tangent to the graph of $y = \ln(x^2 + 1)$.

35*) Suppose $f(x) \leq x \cos\left(\frac{1}{\sqrt{x}}\right)$ for all $x > 0$. Can we compute $\lim_{x \to 0^+} f(x)$?
What if $|f(x)| \leq x \cos\left(\frac{1}{\sqrt{x}}\right)$ for all $x > 0$?

Note that $-x \leq x \cos\left(\frac{1}{\sqrt{x}}\right) \leq x$ and therefore by the squeeze theorem

$$\lim_{x \to 0^+} x \cos\left(\frac{1}{\sqrt{x}}\right) = 0.$$

Similarly

$$\lim_{x \to 0^+} -x \cos\left(\frac{1}{\sqrt{x}}\right) = 0.$$

Now, in the case $|f(x)| \leq x \cos\left(\frac{1}{\sqrt{x}}\right)$, by eliminating the absolute value we get the bounds

$$-x \cos\left(\frac{1}{\sqrt{x}}\right) \leq f(x) \leq x \cos\left(\frac{1}{\sqrt{x}}\right)$$

therefore by the squeeze theorem

$$\lim_{x \to 0^+} f(x) = 0.$$

If we only have $f(x) \leq x \cos\left(\frac{1}{\sqrt{x}}\right)$ we can't apply the same reasoning, because we are missing a lower bound and we can't determine the limit of $f(x)$.

36**) Does there exist a common tangent to the curves $y = e^x$ and $y = -x^2$?
Let's compute the equation of the tangent line to $f(x) = e^x$ at a generic point (a, e^a). $f'(x) = e^x$ so the equation of such tangent is $y - e^a = e^a(x - a)$ and simplifying

$$y = e^a x - ae^a + e^a.$$

Now let's compute the tangent to $g(x) = -x^2$ at a generic point $(b, -b^2)$. $g'(x) = -2x$ so the tangent is $y + b^2 = -2b(x - b)$ and simplifying

$$y = -2bx + b^2.$$

The curves have a common tangent if the slopes and the y-intercepts of the 2 tangents are the same, in other words if the following system admits a solution:

$$\begin{cases} e^a = -2b \\ -ae^a + e^a = b^2 \end{cases} \Rightarrow \begin{cases} b = -\frac{1}{2}e^a \\ -ae^a + e^a = \frac{1}{4}e^{2a} \end{cases}.$$

The system has solution if the equation $-ae^a + e^a = \frac{1}{4}e^{2a}$ has a solution. Dividing both sides by e^a the equation reduces to

$$-a + 1 = \frac{1}{4}e^{2a}$$

which can't be solved explicitly. So we rewrite it as

$$\frac{1}{4}e^{2a} + a - 1 = 0$$

and consider the function $F(a) = \frac{1}{4}e^{2a} + a - 1$ which is continuous everywhere. Moreover $F(0) = \frac{1}{4} - 1 < 0$ and $F(1) = \frac{1}{4}e > 0$ so by the IVT there is at least a solution. Therefore there is at least one common tangent to the 2 graphs.